Int J Theor Phys (2009) 48: 2441–2444
DOI 10.1007/s10773-009-0034-x

An Improved Thin Film Model of the Schwarzschild Black Hole Entropy

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Received: 2 March 2009 / Accepted: 6 May 2009 / Published online: 22 May 2009

Abstract In this paper we improve the thin film model by assuming that there is a limited vicinity outside the black hole horizon. The improved model not only gives a clearer physical meaning of the black hole entropy but also avoids some drawbacks of the original way.

Keywords Entropy · Thin film model · Limited vicinity

Since Bekenstein and Hawking [1–3] proved the entropy of a black hole was proportional to its surface area, there have been many works to study the black hole entropy [4–6]. The brick-wall model [7] proposed by 't Hooft has been extensively studied in connect with the statistical origin of the black hole entropy. The model is devoted to the problem of seeking the statistical origin of the black hole entropy and gives a reasonable explanation. In this model, the statistical property of the external field outside the black hole has been investigated with the brick-wall condition: The field is supposed to vanish near the horizon and at a large distance. The thin film model [8, 9] proposed by Li Xiang and Zhao Zheng improved this model. The model presumes that the Bekenstein-Hawking entropy is associated with the fields in an infinite small region outside of the black hole horizon, so this model can be used to nonstatic black holes. As the first model of nonstatic black hole entropy, this method has been applied to the various black hole models [10–13] and the result is satisfactory.

Although the thin film model is popular, it has three drawbacks.

(1) Though the brick-wall model has the quantum correction to black hole entropy with a logarithmic divergent term, after the approximation, the result of the thin film model is the same as classical result without quantum correction.
(2) During the calculation on the thin film model, there are two infinitesimal terms simultaneously. Their physical meaning is ambiguous.
(3) The cutoff in this model is like more a mathematical choice than a physical one.

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To treat these problems, we improve the model. In our opinion, the Bekenstein-Hawking entropy should be associated with the fields in a limited film near the black hole horizon instead of an infinite small film. After the calculation

(1) We get the quantum correction while it can be used to nonstatic black holes.
(2) There is only an infinitesimal term through the calculation.
(3) We can give the cutoff a clearer physical interpretation in this model.

Here we introduce our idea by the Schwarzschild black hole.

The line element of the Schwarzschild black hole is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \tag{1}$$

where M is the mass of the black hole. The position of event horizon is

$$r_h = 2M \tag{2}$$

and the temperature of the horizon is

$$T = \frac{1}{\beta} = \frac{1}{8\pi M} \tag{3}$$

Considering (1), we can write the Klein-Gordon equation for scalar field

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right)\phi = m_0^2\phi \tag{4}$$

as

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial x^\mu}\left(r^2\sin\theta g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right)\phi - m_0^2\phi = 0 \tag{5}$$

Due to the spherical symmetry of the Schwarzschild spacetime, we can separate the variables by writing $\phi(r,\theta,\varphi,t) = f(r)Y_{lm}(\theta,\varphi)\exp(-iEt)$, the equation of radial part can be written as

$$\left(1 - \frac{2M}{r}\right)^{-1} E^2 f(r) + \frac{1}{r}\frac{d}{dr}\left[r^2\left(1 - \frac{2M}{r}\right)\frac{d}{dr}f(r)\right] - \left(m_0^2 + \frac{l(l+1)}{r^2}\right)f(r) = 0 \tag{6}$$

Supposing the expression $f(r) = \exp(iS(r))$, we can get the radial momentum k_r by WKB approximation

$$k_r^2 = \left(\frac{\partial S(r)}{\partial r}\right)^2 = \left[\left(1 - \frac{2M}{r}\right)^{-1} E^2 - \left(m_0^2 + \frac{l(l+1)}{r^2}\right)\right]\left(1 - \frac{2M}{r}\right)^{-1} \tag{7}$$

According to the theory of canonical ensemble, the free energy reads

$$\beta F = \sum_E \ln(1 - e^{-\beta E}) \tag{8}$$

In terms of the semiclassical theory and assuming that the energy E is continuous, we replace the sum by integration

$$\beta F = \int_0^{+\infty} dE g_{(E)} \ln(1 - e^{-\beta E}) = \int_0^{+\infty} d\Gamma_{(E)} \ln(1 - e^{-\beta E})$$

$$= -\beta \int_0^{+\infty} \frac{\Gamma_{(E)}}{e^{\beta E} - 1} dE \tag{9}$$

where $\Gamma_{(E)}$ is the total number of microscopic state with energy less than E, $g_{(E)} = d\Gamma_{(E)}/dE$ is the density of states.

According to semiclassical quantization rule, we have

$$\begin{aligned}\Gamma_{(E)} &= \sum_{lm} n_r(E, l, m) = \sum_{l} (2l+1) n_r(E, l) \\ &= \int_l (2l+1) dl \frac{1}{\pi} \int_r k_r(r, E, l) dr\end{aligned} \tag{10}$$

Therefore, the expression of free energy (9) can be written as

$$F = -\frac{1}{\pi} \int_0^{+\infty} dE \int_r dr \int_l (2l+1) dl \frac{k_r(r, E, l)}{e^{\beta E} - 1} \tag{11}$$

Substituting radial momentum into (11), we have

$$\begin{aligned}F = &-\frac{1}{\pi} \int_0^{+\infty} dE \int_r dr \int_l (2l+1) dl (e^{\beta E} - 1)^{-1} \\ &\times \left\{ \left[\left(1 - \frac{2M}{r}\right)^{-1} E^2 - \left(m_0^2 + \frac{l(l+1)}{r^2}\right) \right] \left(1 - \frac{2M}{r}\right)^{-1} \right\}^{\frac{1}{2}}\end{aligned} \tag{12}$$

The upper limit of the integration with respect to l is taken so that k_r^2 is non-negative, and the lower limit is naturally zero, after the integration of l

$$F = -\frac{2}{3\pi} \int_0^{+\infty} \frac{dE}{e^{\beta E} - 1} \int_r dr \frac{r^2}{(1 - 2M/r)^2} \left[E^2 - \left(1 - \frac{2M}{r}\right) m_0^2 \right]^{3/2} \tag{13}$$

Different from a large length L used by the brick-wall model and a thin film used by the thin film model, we assume that the black hole entropy attributes to a limited vicinity outside the black hole horizon. So the integration of r should be calculated on the region $(r_h + \epsilon, r_h + a)$

$$F = -\frac{2}{3\pi} \int_0^{+\infty} \frac{dE}{e^{\beta E} - 1} \int_{r_h+\epsilon}^{r_h+a} dr \frac{r^2}{(1 - 2M/r)^2} \left[E^2 - \left(1 - \frac{2M}{r}\right) m_0^2 \right]^{3/2} \tag{14}$$

When a is very small, the coefficient with respect to m_0 of the last part is approximation to zero. We get

$$\begin{aligned}F &= -\frac{2}{3\pi} \int_0^{+\infty} \frac{E^3 dE}{e^{\beta E} - 1} \int_{r_h+\epsilon}^{r_h+a} dr \frac{r^2}{(1 - 2M/r)^2} \\ &= -\frac{2}{3\pi} \frac{\pi^4}{15\beta^4} \int_{r_h+\epsilon}^{r_h+a} r^2 \left(1 - \frac{2M}{r}\right)^{-2} dr \\ &= -\frac{2\pi^3}{45\beta^4} \left[\frac{1}{3}(a^3 - \epsilon^3) + 4M(a^2 - \epsilon^2) + 24M^2(a - \epsilon) \right. \\ &\quad \left. + 32M^3(\ln a - \ln \epsilon) + 16M^4 \left(\frac{1}{\epsilon} - \frac{1}{a} \right) \right]\end{aligned} \tag{15}$$

and get rid of the divergent term and infinite small terms. The first four terms correspond to the quantum correction. And the last term corresponds to the classical result. Only using the last term, we have

$$F = \frac{4\pi^3}{45\beta^4}\frac{(2M)^4}{a} \tag{16}$$

If we let $a = T/45 = 1/(45 \times 8\pi M) = 1/(360\pi M)$, we can get the entropy

$$S = \beta^2 \frac{\partial F}{\partial \beta} = \frac{256\pi^3 M^4}{45\beta^3 a} = \frac{A_h}{4} \tag{17}$$

here A_h is the area of the black hole horizon. It is the same as the classical Bekenstein-Hawking entropy.

Here we get the cutoff $1/(360\pi M)$. In our opinion, it is just giving the thickness of the film near the black hole horizon. It responds with the conclusion that the maximum of the Planckian emission spectrum of Hawking's thermal radiation is on the order of $1/M$ magnitude [14].

In summary, by using the assumption of a limited vicinity outside the black hole horizon, the statistical entropy of Schwarzschild black hole is calculated in this report. The improved model not only gives a clearer physical meaning but also avoids some drawbacks of the original way.

Acknowledgements The Project was Sponsored by Natural Science Foundation of Hunan Province under Grant No.06JJ20026 and by scientific Research Fund of Hunan Provincial Education Department.

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